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Existence of a Phase Transition in a Continuous Quantum System

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We prove the existence of a phase transition in the quantum Widom–Rowlison model in two dimension. The case of Boltzmann and Bose statistics are explicitly discussed.

KEY WORDS: Phase transitions; continuous quantum systems; Peierls argument; Feynmann–Kac formula.

1. INTRODUCTION

In 1971, Ruelle[(4)] proved the existence of a phase transition for the classical Widom and Rowlison model.[(5)] This is the first and, until recently,[(3)] the only continuous model for which the existence of a phase transition has been rigorously proven. In this paper we show that it is possible to extend Ruelle's proof to the quantum version of the Widom and Rowlison model.

We will describe in some detail the simpler case of quantum particles obeying the Boltzmann Statistics and then show how these arguments apply also to Bose particles.

For sake of definiteness we will consider the two dimensional case. Higher dimensions can be handled in a similar manner.

As it is well known, Feynmann–Kac formula provides a representation of the operator $\exp(-tH)$ for $t > 0$ where H is the Hamiltonian of a system of quantum particles, as a integral of a functional of the potentials

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over the sample functions of a Wiener Process. Our work is based on the application of this formalism to Statistical Mechanics as discussed in Ginibre.[1]

The model we consider describes a gas of two kind of particles, say A and B. The only interaction is a hard-core exclusion between unlike particles i.e., an A-particle and a B-particle cannot be closer than a distance R, there is no restriction on the distance between two A or two B-particles. If the activity of the two kinds of particles are set equal to z, where $z=\zeta/\beta$, β is the inverse temperature and $\mu=\frac{1}{\beta}\log\zeta$ the chemical potential, when z is sufficiently large, one expects at least two coexisting phases: one predominantly of A-particles and another predominantly of B-particles.

In the classical case, Ruelle uses a suitable modification of the Peierls argument to prove that is is actually the case.

We briefly summarize the main steps of Ruelle's proof:

a) The volume Λ containing the particles is considered as composed of N little boxes of size smaller that R, the hard core range.

b) Each box, due to the hard core condition, can contain only alike particles, so that shading the boxes occupied by one kind of particles, say B-particles, it is possible to make a partition of the set of microscopic configurations into subsets corresponding to the possible distribution of the darker spots on the lattice of the N boxes composing the volume Λ.

c) Suitable self avoiding contours are defined and their probability is computed by using the factorization properties of the contributions coming from compatible contours and the symmetry between A and B-particles.

d) It is then possible to check that, at sufficiently large activities, Peierls estimate holds.

The difficulty introduced by the quantum nature of the particles is that, in the Ginibre formulation,[1] a microscopic configuration is not described any more by a set of points but by a set of random loops. Even in this case it is possible to define self avoiding contours describing the B-particles configurations, but the factorization properties do not hold anymore because the loops associated to A-particles can be arbitrarily large.

We will show that when the temperature is sufficiently high, since the loops tend to shrink and, typically, became very small, it is possible to get an upper bound for the probability of a contour very similar to the classical one and then use the Peierls argument.

We remark that, both in the classical and the semi-classical case, $z=\zeta/\beta$ is the parameter that controls the average density and therefore in both cases we can get large densities for ζ fixed (e.g., $\zeta < 1$ as required by

Bose statistics for ideal gases) and β sufficiently small. However in the Boltzmann case, one would expect that the phase transition actualy occurs for any β provided ζ is sufficiently large but in our approach we are unable to prove it.

2. DEFINITION OF THE CONTOURS AND STATEMENTS OF THE MAIN RESULT

Following Ginibre,[1] the partition function of our model in a finite volume $\Lambda \subset \mathbb{R}^2$ is given by

$$Z_{\Lambda,\beta,\zeta} \equiv \sum_{n_1,n_2=0}^{\infty} \frac{\zeta^{n_1}}{n_1!}\frac{\zeta^{n_2}}{n_2!} \int_{\Lambda^{n_1}} dx_1 \cdots dx_{n_1} \prod_{i=1}^{n_1} \mathbb{P}^{\beta}_{x_i}[d\omega_{x_i}]\, \alpha_\Lambda(\omega_{x_i}) \int_{\Lambda^{n_2}} dy_1 \cdots dy_{n_2}$$
$$\times \prod_{j=1}^{n_2} \mathbb{P}^{\beta}_{x_j}[d\omega_{x_j}]\, \alpha_\Lambda(\omega_{y_j}) \prod_{i,j}^{n_1,n_2} e^{-\int_0^\beta dt\, U(\omega_{x_i}(t),\, \omega_{y_j}(t))} \tag{2.1}$$

here $\mathbb{P}^{\beta}_{x}(d\omega)$ is the constrained Wiener measure (unormalized), where ω is a continuous function from $[0, \beta]$ to $\mathbb{R}^2$, with $\omega(0) = \omega(\beta) = x$. Remark that $\mathbb{P}^{\beta}_{x}(d\omega)$ is a positive measure but not a probability, since $\int \mathbb{P}^{\beta}_{x}(d\omega) = (2\pi\beta)^{-1}$.

$$\alpha_\Lambda(\omega) = \begin{cases} 1, & \text{if } \quad \forall t \in [0, \beta],\, \omega \in \Lambda \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

and

$$U(\omega_{x_i}(t), \omega_{y_j}(t)) = \begin{cases} +\infty, & \text{if } \quad |\omega_{x_i}(t) - \omega_{y_j}(t)| < R \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

is the hard core potential. In a more compact notation, we will write

$$Z_{\Lambda,\beta,\zeta} = \int dX_\Lambda\, dY_\Lambda\, \alpha_\Lambda(X_\Lambda)\, \alpha_\Lambda(Y_\Lambda)\, h(X_\Lambda, Y_\Lambda) \tag{2.4}$$

where $h(X_\Lambda, Y_\Lambda)$ is the hard core condition, that is the last factor in (2.1), and

$$\int dX_\Lambda \equiv \sum_{n_1=0}^{\infty} \frac{\zeta^{n_1}}{n_1!} \int_{\Lambda^{n_1}} dx_1 \cdots dx_n \prod_{i=1}^{n_1} \mathbb{P}^{\beta}_{x_i}[d\omega_{x_i}] \tag{2.5}$$

Given $n_1, n_2, x_1, ..., x_{n_1}, y_1, ..., y_{n_2}$ a configuration

$$\underline{\omega}_\Lambda \equiv (\underline{\omega}^A_\Lambda, \underline{\omega}^B_\Lambda) \tag{2.6}$$

is a set of closed trajectories $\omega^A_{x_1}, ..., \omega^A_{x_{n_1}}, \omega^B_{y_1}, ..., \omega^B_{y_{n_2}}$ compatible with the hard core condition and fully contained in Λ.

Following Ruelle,[4] we assume that Λ is such that we can make a partition of Λ in boxes $\mathscr{D}$ of side $d = R/3\sqrt{2}$ and given a configuration $\underline{\omega}_\Lambda$, for each $\omega^B_{y_i}(t)$ we consider the boxes visited by the trajectory and shade the $3d \times 3d$ squares centered around them.

The boundary of the union of the shaded areas is a contour Γ, i.e., a polygonal consisting of various connected components. We decompose Γ as an union of disjoint pieces $\gamma_1, ..., \gamma_n$. Each piece is a R-connected component, that is the smallest set of connected components of Γ such that if two connected components have a distance less than R, they belong to the same γ_i.

We introduce the boundary condition that non B-trajectory is allowed to reach the last two rows and columns adjacent to the sides of Λ.

We will consider an external contour γ, i.e., we can construct a continuous path in $\mathbb{R}^2$ coming from infinity that reaches a point of γ without crossing Γ.

If we call $\mathbb{1}(X_\Lambda, Y_\Lambda; \gamma)$ the indicator function of the configurations giving rise to γ as an external contour, then the grand canonical probability of the contour γ is

$$\begin{aligned}\mu_{\Lambda, \beta, \zeta}(\gamma) &\equiv \frac{1}{Z_{\Lambda, \beta, \zeta}} \int dX_\Lambda \int dY_\Lambda \, \alpha_\Lambda(X_\Lambda) \, \alpha^*_\Lambda(Y_\Lambda) \, h(X_\Lambda, Y_\Lambda) \, \mathbb{1}(X_\Lambda, Y_\Lambda; \gamma) \\ &\equiv \frac{Z_{\Lambda, \beta, \zeta}(\gamma)}{Z_{\Lambda, \beta, \zeta}} \end{aligned} \tag{2.7}$$

where $\alpha^*_\Lambda(Y_\Lambda)$ takes into account the boundary conditions.

Our main result is

Theorem 2.1. There exists a β_0 such that for all Λ, for all $\beta < \beta_0$, and for all $\zeta > 0$

$$\mu_{\Lambda, \beta, \zeta}(\gamma) \leqslant e^{-\zeta \, |\gamma| \, (d^2/64\pi\beta)} \tag{2.8}$$

here $|\gamma| \, d$ is the length of the polygonal γ.

The proof of the existence of a phase transition is now almost identical to the one of the classical case and for sake of completeness is given in Appendix 2.

Remark. In the theorem ζ and β are independant parameters and we need only a condition on β to prove it. On the other hand, the relevant parameter to prove the existence of a phase transition (cf. Appendix 2) is ζ/β and therefore in that case we will get a $\beta_0(\zeta)$.

Remark. If we notice that the dimensionless parameter that controls our estimates is $d/\sqrt{\beta}=R/(3\sqrt{2\beta}) \gg 1$, our result is obvious from a physical point of view: the classical estimate holds when the de Broglie thermal wavelength is much smaller than the range of the potential.

3. THE PROOF

We start describing the heuristics of the proof. It follows from the shading procedure and the definition of a R-connected external contour that in presence of a γ external contour the B-trajectories are strictly contained in two disconnected regions C_1 and C_2, where C_1 is the region internal to γ and C_2 is the region outside γ, where the B-particles can travel without producing contours closer than R to γ. Therefore

$$Z_{\Lambda,\beta,\zeta}(\gamma)=\int dX_\Lambda\, \alpha_\Lambda(X_\Lambda) \int dY_{C_2}\, \tilde{\alpha}^*_{C_2}(Y_{C_2}) \int dY_{C_1}\, \tilde{\alpha}_{C_1}(Y_{C_1})\, h(X_\Lambda, Y_{C_1} \cup Y_{C_2}) \tag{3.1}$$

where $\alpha^*_{C_2}(Y_{C_2})$ and $\tilde{\alpha}_{C_1}(Y_{C_1})$ take into account the external boundary conditions and the extra constraints that γ exists and is an external component, i.e., all B-trajectories outside γ cannot get closer than d to the boundary of C_2 and B-trajectories inside γ have to shade at least the band touching the internal boundary of the contour.

If we set

$$Z_1^B \equiv \int dX_{C_1^-}\, dY_{C_1}\, \alpha_{C_1^-}(X_{C_1^-})\, \tilde{\alpha}_{C_1}(Y_{C_1})\, h(X_{C_1^-}, Y_{C_1}) \tag{3.2}$$

$$Z_1^A \equiv \int dY_{C_1^-}\, dX_{C_1}\, \alpha_{C_1^-}(Y_{C_1^-})\, \tilde{\alpha}_{C_1}(X_{C_1})\, h(Y_{C_1^-}, X_{C_1}) \tag{3.3}$$

where C_1^- is the region (if any) internal to γ at a distance larger than R from γ.

$$Z_2^A \equiv \int dX_{C_2}\, dY_{C_2}\, \alpha_{C_2}(X_{C_2})\, \alpha^*_{C_2}(Y_{C_2})\, h(X_{C_2}, Y_{C_2}) \tag{3.4}$$

and, calling G the band outside γ made of little squares which have one side or corner on γ,

$$\tilde{Z}_G^A \equiv \int dX_G\, \alpha_G(X_G) = \exp\left(\zeta \int_G dx\, \mathbb{P}_x^\beta[d\omega_x]\, \alpha_G(\omega_x)\right) \tag{3.5}$$

(the last equality follows from the fact that the A-trajectories are independent), we get

$$Z_{\Lambda,\beta,\zeta} \geqslant Z_1^A Z_2^A \tilde{Z}_G^A \tag{3.6}$$

since the A-trajectories inside G are at distance larger that R from all B-trajectories contained in Z_1^A and Z_2^A.

Now if we ignore

1) The A-trajectories starting in C_1^- (resp. C_2) that do cross the boundary of C_1^- (resp. C_2).

2) All the trajectories starting in $\Lambda \setminus (C_1^- \cup C_2)$.

Formula (3.1) becomes

$$Z_{\Lambda,\beta,\zeta}(\gamma) = Z_1^B Z_2^A \tag{3.7}$$

and since by symmetry $Z_1^A = Z_1^B$, we get

$$\begin{aligned} \mu_{\Lambda,\beta,\zeta}(\gamma) &\leqslant \frac{1}{\tilde{Z}_G^A} \\ &\leqslant \exp\left(-\zeta \int_G dx\, \mathbb{P}_x^\beta[d\omega_x]\, \alpha_G(\omega_x)\right) \\ &\leqslant e^{-\zeta/8\pi\beta\, |\gamma|\, d^2} \end{aligned} \tag{3.8}$$

The last inequality follows from the properties of the Wiener trajectories when β is sufficiently small cf. Appendix 1 Eq. (5.9).

Unfortunately hypothesis 1) and 2) are not true, because the hard core condition holds only at equals times and for a given generic configuration of B-trajectories, an A-trajectory can go almost everywhere in Λ, if it carefully avoids the sinchronic approach of the B-trajectories that are present.

In our proof we will show that trajectories described by conditions 1) and 2) can be neglected for β small enough.

In fact, a property of the conditional Wiener measure is that, for any $\delta > \sqrt{2\beta}$,

$$\int \mathbb{P}_x^\beta(d\omega)\, \mathbb{1}_{\{\exists t \in (0,\beta) : d(\omega(t), x) > \delta\}} \leqslant \frac{e^{-\delta^2/2\beta}}{2\pi\beta} \tag{3.9}$$

cf. Appendix 1 Eq. (5.7). So that when β is sufficiently small, the probability for a trajectory starting in C_1^- (resp. C_2) to cross the boundary of C_1^- (resp. C_2) becomes very small.

The same argument applies also to a trajectory starting in $\Lambda \setminus (C_1^- \cup C_2)$. In fact, $\forall x \in \mathscr{D} \subset \Lambda \setminus (C_1^- \cup C_2)$, $\exists t_x^*(\underline{\omega}^B) \in [0, \beta]$ such that $d(\omega_x(t_x^*), \mathscr{D}) > 0$. That is a Wiener trajectory, whose end points belong to a generic little box $\mathscr{D}$ contained in $\Lambda \setminus (C_1^- \cup C_2)$, cannot stay forever in the same little box, because sooner or later a B-trajectory (creating the contour γ) will get closer than R.

We start by making a finer partition of Λ that will allows to define in a more precise way the subset of the A-trajectories that we are going to neglect.

Definition 3.1. $\Lambda(\gamma)$ is the region in Λ, outside γ, where the B-particles can travel without producing any contour closer than R to γ.

Definition 3.2. $C(\gamma)$ is the region in Λ, inside γ.

Note that the region bounded by an external contour can be decomposed in connected components each of which is simply connected.

Definition 3.3. For $\delta > 0$, let

$$G(\gamma, -\delta) \equiv \{x \in C(\gamma) : d(x, \gamma) \leqslant \delta\} \tag{3.10}$$

That is the part of the δ-neighborhood of γ which is inside $C(\gamma)$.

Definition 3.4.

$$G(\gamma, \delta) \equiv \{x \notin C(\gamma) : d(x, \gamma) \leqslant \delta\} \tag{3.11}$$

That is the part of the δ-neighborhood of Γ which is outside $C(\gamma)$.

To simplify some formulae to come, let us denote

$$\begin{aligned}
C_2 &\equiv \Lambda(\gamma)\\
C_2^+ &\equiv \Lambda \setminus \left(C(\gamma) \cup G\left(\gamma, \frac{d}{2}\right)\right)\\
C_2^- &\equiv \Lambda \setminus (C(\gamma) \cup G(\gamma, d))\\
C_1 &\equiv C(\gamma)\\
C_1^+ &\equiv C(\gamma) \setminus G\left(\gamma, -R+\frac{d}{2}\right)\\
C_1^- &\equiv C(\gamma) \setminus G(\gamma, -R)
\end{aligned}$$

$$
\begin{aligned}
C_3 &\equiv \Lambda \setminus (C_1^- \cup C_2^-) \\
G^+ &\equiv G(\gamma, d) \\
G^- &\equiv G(\gamma, d) \setminus G\left(\gamma, \frac{d}{2}\right)
\end{aligned}
\tag{3.12}
$$

Note that C_1^-, C_1^+ can be empty.

Let D be a region contained in Λ, we denote

$$
\mathscr{A}(D, \delta) \equiv \{\omega_x, x \in D : \exists t(x) \in (0, \beta), d(\omega_x(t(x)), D) > \delta\} \tag{3.13}
$$

Using these definitions we can now write an upper bound for the numerator of (2.7) in a way that singles out the contributions coming from the A-trajectories:

i) whose endpoints belong to C_1^- (resp. C_2^-) but that do cross the boundaries of C_1^+ (resp. C_2^+).

ii) whose endpoints belong to $\Lambda \setminus (C_2^- \cup C_1^-) = C_3$.

We these notations, we have:

$$
\begin{aligned}
Z_{\Lambda, \beta, \zeta}(\gamma) \leqslant{}& \tilde{Z}_1^B \tilde{Z}_2^A \left(1 + \int dX_{C_2^-} \mathbb{1}_{\{\mathscr{A}(C_2^-, d/2)\}}(X_{C_2^-})\right) \\
&\times \left(1 + \int dX_{C_1^-} \mathbb{1}_{\{\mathscr{A}(C_1^-, d/2)\}}(X_{C_1^-})\right) \left(1 + \int_{\mathscr{D}} dX_{\mathscr{D}} \mathbb{1}_{\{\mathscr{A}(\mathscr{D}, d)\}}(X_{\mathscr{D}})\right)^{2|C_3|/|\mathscr{D}|}
\end{aligned}
\tag{3.14}
$$

where

$$
\tilde{Z}_1^B \equiv \int dY_{C_1}\, dX_{C_1^-}\, \tilde{\alpha}_{C_1}(Y_{C_1})\, \alpha_{C_1^+}(X_{C_1^-})\, h(Y_{C_1}, X_{C_1^-}) \tag{3.15}
$$

and

$$
\tilde{Z}_2^A \equiv \int dX_{C_2^-}\, dY_{C_2}\, \alpha_{C_2^+}(X_{C_2^-})\, \alpha^*_{C_2}(Y_{C_2})\, h(X_{C_2^-}, Y_{C_2}) \tag{3.16}
$$

On the other hand

$$
Z_{\Lambda, \beta, \zeta} \geqslant \tilde{Z}_1^A \tilde{Z}_2^A \tilde{Z}_{G^-}^A \tag{3.17}
$$

where $\tilde{Z}_{G^-}^A$ is defined as in (3.5) and

$$
\tilde{Z}_1^A \equiv \int dX_{C_1}\, dY_{C_1^-}\, \tilde{\alpha}_{C_1}(X_{C_1})\, \alpha_{C_1^+}(Y_{C_1^-})\, h(X_{C_1}, Y_{C_1^-}) \tag{3.18}
$$

therefore, we get

$$\mu_{\Lambda,\beta,\zeta}(\gamma) \leqslant \frac{e^{\zeta(2\,|C_3|/|\mathscr{D}|\int_{\mathscr{D}} dx\, \mathbb{P}_x^\beta[d\omega]\, \mathbb{1}_{\{\mathscr{A}(\mathscr{D},\,d)\}}(\omega) + \int_{C_2^-} dx\, \mathbb{P}_x^\beta[d\omega]\, \mathbb{1}_{\{\mathscr{A}(C_2^-,\,d/2)\}}(\omega) + \int_{C_1^-} dx\, \mathbb{P}_x^\beta[d\omega]\, \mathbb{1}_{\{\mathscr{A}(C_1^-,\,d/2)\}}(\omega))}}{e^{\zeta\int_{G^-} dx\, \mathbb{P}_x^\beta[d\omega]\, \alpha_{G^-}(\omega)}} \tag{3.19}$$

Since, for any $\delta > \sqrt{2\beta}$

$$\int_D dx\, \mathbb{P}_x^\beta[\omega]\, \mathbb{1}_{\mathscr{A}(D,\,\delta)}(\omega) \leqslant \frac{|\partial D|}{\sqrt{2\pi\beta}}\, e^{-\delta^2/2\beta} \tag{3.20}$$

and

$$\int_{G^-} dx\, \mathbb{P}_x^\beta[d\omega]\, \alpha_{G^-}(\omega) \geqslant \frac{|G^-|}{8\pi\beta} \tag{3.21}$$

cf. Appendix 1, if we note that $|\partial G^+|$, $|\partial C_2^-|$, $|\partial C_1^-|$, and $|C_3|$ are all bounded by a constant times $|G^-|$ we get that, for β sufficiently small,

$$\mu_{\Lambda,\beta,\zeta}(\gamma) \leqslant e^{-\zeta\,|G^-|/16\pi\beta} \tag{3.22}$$

since $|G^-| = |\gamma|\, d^2/4$, this entails (2.8).

4. THE BOSE GAS

For identical quantum particles, the grand canonical partition function (see ref. 1) reads

$$\begin{aligned} Z_{\Lambda,\beta,\zeta} = {} & \sum_{n_1,\,n_2=0}^{\infty} \int_{\Lambda^{n_1}} dx_1 \cdots dx_{n_1} \int_{\Lambda^{n_2}} dy_1 \cdots dy_{n_2} \\ & \times \prod_{\ell=1}^{n_1} \prod_{m=1}^{n_2} \sum_{j_\ell=1}^{\infty} \sum_{j_m=1}^{\infty} \left(\frac{\zeta^{j_\ell}}{j_\ell}\, \varepsilon^{j_\ell-1} \mathbb{P}_{x_\ell}^{j_\ell\beta}[d\omega_{x_\ell}^A]\, \frac{\zeta^{j_m}}{j_m}\, \varepsilon^{j_m-1} \mathbb{P}_{y_m}^{j_m\beta}[d\omega_{y_m}^B]\, \alpha_\Lambda(\underline{\omega}^A, \underline{\omega}^B) \right. \\ & \left. \times \prod_{k_\ell=1}^{j_\ell} \prod_{k_m=1}^{j_m} e^{-\int_0^\beta U(\omega_{x_\ell}(t+(k_\ell-1)\,\beta),\, \omega_{y_m}(t+(k_m-1)\,\beta))\, dt} \right) \end{aligned} \tag{4.1}$$

where $\varepsilon = 1$ for Bosons and $\varepsilon = -1$ for Fermions.

Therefore, in the case of a Bose gas, the measure associated to a trajectory ω_z (for $z=x, y$) is now

$$\sum_{j=1}^{\infty} \frac{\zeta^j}{j} \mathbb{P}_z^{j\beta}[d\omega_z] \tag{4.2}$$

and following without any change, the same steps of the previous section, we get, for $\zeta \leqslant 1$

$$\begin{aligned} \mu_{\Lambda, \beta, \zeta}(\gamma) &\leqslant e^{-\int_{G^-} dx \sum_{j=1}^{\infty} \zeta^j/j\, \mathbb{P}_x^{j\beta}[d\omega_x]\, \alpha_{G^-}(\omega_x)} \\ &\leqslant e^{-(|\gamma|\, d^2/64\pi\beta)\, g(\zeta)} \end{aligned} \tag{4.3}$$

where $g(\zeta) \equiv \sum_{j=1}^{\infty} \zeta^j/j^2$ and therefore we recover a bound similar to (2.8).

Remark. We note that our result makes sense even at $\zeta=1$. Since our method works in a similar way in higher dimension and give the same constraints on ζ, a corresponding result holds also in 3 and more dimension, when the Bose condensation takes place for the A free gas. However this does not imply that the condensation phenomenon is present also in the Widom–Rowlison model.

In fact in a finite volume Λ the model make sense for $\zeta < e^{\beta e_0(\Lambda)}$ where $e_0(\Lambda)$ is the lowest eigenvalue of the Laplace operator with Dirichlet boundary conditions on Λ.

Since for $|\Lambda|=L^\nu$, $e_0(\Lambda)=\text{const.}/L^2 > 0$, the critical value is $\zeta_c = e^{\beta\, \text{const.}/L^2} > 1$ and for any finite volume Λ, $\zeta=1$ is strictly below this value. It follows that, with our estimates, it is impossible to reach the range of values of ζ relevant for the description of the condensation phenomenon. The study of the condensation phenomena for this model is by no means trivial, see for instance ref. 2.

Remark. In the case of a Fermi gas the weight associated to a trajectory is not anymore positive definite and some extra work is needed in order to disentangle the relevant contributions and get the bound.

If we consider the following identity

$$\int dX_\Lambda\, \alpha_\Lambda(X_\Lambda) = \int dX_C\, dX_{\Lambda\setminus C}\, \alpha_C(X_C)\, \alpha_{\Lambda\setminus C}(X_{\Lambda\setminus C})[1 + A_C + A_{\Lambda\setminus C} + A_C A_{\Lambda\setminus C}] \tag{4.4}$$

where

$$A_D \equiv \frac{\int dX_D(1-\alpha_D(X_D))}{dX_D\, \alpha_D(X_D)} \tag{4.5}$$

and remark that $\int dX_D\, \alpha_D(X_D) > 0$, being the trace of a positive operator, and that

$$|A_D| = \frac{|\int dX_D(1-\alpha_D(X_D))|}{\int dX_D\, \alpha_D(X_D)} \leqslant e^{-(|D|/16\pi\beta)\, g_F(\zeta)} \tag{4.6}$$

where $g_F(\zeta) = \sum_{j=1}^{\infty} (-1)^{j-1} \zeta^j/j^2$, it should be possible to preserve the positivity of the relevant contributions and recover the bound (2.8) even in this case.

5. APPENDIX 1

In this section we prove (3.20) that is, for any $\delta > \sqrt{2\beta}$ and any volume D

$$\int_D dx\, \mathbb{P}_x^\beta[\omega]\, \mathbb{1}_{\mathscr{A}(D,\delta)}(\omega) \leqslant \frac{|\partial D|}{\sqrt{2\pi\beta}}\, e^{-\delta^2/2\beta} \tag{5.1}$$

where

$$\mathscr{A}(D,\delta) \equiv \{\omega_x : x \in D, \exists t(x) \in (0,\beta), d(\omega_x(t(x)), D) > \delta\} \tag{5.2}$$

recall that $\mathbb{P}_x^\beta[\omega]$ is the Wiener measure conditioned to start in x at $t=0$ and end at x at time β, that is a two dimensional Brownian "Bridge." We first note that

$$\mathscr{A}(D,\delta) \subset \{\omega_x : x \in D, \exists t(x) \in (0,\beta), d(\omega_x(t(x)), x) > d(x, \partial D) + \delta\} \tag{5.3}$$

On the other hand, we note that, by discretizing the interval of time $[0,\beta]$, we have, for any $r > 0$,

$$\begin{aligned}\int \mathbb{P}_x^\beta[d\omega_x]\, \mathbb{1}_{\{\exists t(x) \in [o,\beta]\,:\, d(\omega_x(t_x), x) > r\}} \\ = \lim_{n \uparrow \infty} \sum_{j=1}^{n-1} \mathbb{P}_x^\beta\left[\forall i < j, d\left(\omega_x\left(\frac{i\beta}{n}\right), x\right) < r, d\left(\omega_x\left(\frac{j\beta}{n}\right), x\right) > r\right]\end{aligned} \tag{5.4}$$

To estimate a single term in the previous sum, we write it explicitly, using the notation,

$$\gamma(t,x) \equiv \frac{e^{-\|x\|^2/2t}}{2\pi t} \tag{5.5}$$

where $x \in \mathbb{R}^2$, we have

$$\begin{aligned}
&\mathbb{P}_x^\beta\left[\forall i < j, d\left(\omega_x\left(\frac{i\beta}{n}\right), x\right) < r, d\left(\omega_x\left(\frac{j\beta}{n}\right), x\right) > r\right] \\
&\quad = \int dy_1 \cdots dy_{n-1}\, \mathbb{1}_{\{|y_j - x| > r\}} \prod_{i=1}^{j-1} \mathbb{1}_{\{|y_i - x| < r\}} \\
&\qquad \times \gamma\left(\frac{\beta}{n}, x - y_1\right)\left(\prod_{i=1}^{n-2} \gamma\left(\frac{\beta}{n}, y_i - y_{i+1}\right)\right)\gamma\left(\frac{\beta}{n}, y_{n-1} - x\right) \\
&\quad = \int dy_1 \cdots dy_j\, \mathbb{1}_{\{|y_j - x > r\}} \prod_{i=1}^{j-1} \mathbb{1}_{\{|y_i - x| < r\}} \\
&\qquad \times \gamma\left(\frac{\beta}{n}, x - y_1\right)\left(\prod_{i=1}^{j-2} \gamma\left(\frac{\beta}{n}, y_i - y_{i+1}\right)\right)\gamma\left(\beta\left(1 - \frac{j}{n}\right), y_j - x\right) \\
&\quad \leqslant \gamma\left(\beta\left(1 - \frac{j}{n}\right), r\right) \mathbb{P}\left[\forall i < j, \left|\omega\left(\frac{\beta i}{n}\right) - x\right| < r, \left|\omega\left(\frac{\beta j}{n}\right) - x\right| > r\right]
\end{aligned} \tag{5.6}$$

where the last factor is the standard two dimensional Wiener measure (i.e., a probability measure).

Since we have a partition, for all $r > \sqrt{2\beta}$, we get

$$\begin{aligned}
&\int \mathbb{P}_x^\beta[d\omega_x]\, \mathbb{1}_{\{\exists t(x) \in [o, \beta] : d(\omega_x(t_x), x) > r\}} \\
&\qquad \leqslant \max_{1 \leqslant j \leqslant n-1} \gamma\left(\beta\left(1 - \frac{j}{n}\right), r\right) \leqslant \frac{1}{2\pi\beta} e^{-r^2/2\beta}
\end{aligned} \tag{5.7}$$

where the last step follows from a simple computation.

Therefore collecting (5.3), and (5.7) we get

$$\begin{aligned}
\int_D dx\, \mathbb{P}_x^\beta[\omega]\, \mathbb{1}_{\mathscr{A}(D, d)}(\omega) &\leqslant \frac{1}{2\pi\beta} e^{-r^2/2\beta} \int_D dx\, e^{-d(x, \partial D)/2\beta} \\
&\leqslant \frac{|\partial D|}{\sqrt{2\pi\beta}} e^{-r^2/2\beta}
\end{aligned} \tag{5.8}$$

that is (5.1).

Now we prove (3.21) that is, for β sufficiently small,

$$\int_D dx\, \mathbb{P}_x^\beta[d\omega]\, \alpha_D(\omega) \geqslant \frac{|D|}{8\pi\beta} \tag{5.9}$$

Note that for all $D' \subset D$ we have

$$\int_D dx\, \mathbb{P}_x^\beta[d\omega_x]\, \mathbb{1}_{\{\forall t \in (o,\beta);\, \omega \in D\}}$$

$$\geqslant \int_{D'} dx\, \mathbb{P}_x^\beta(d\omega_x) - \int_{D'} dx\, \mathbb{P}_x^\beta(d\omega_x)\, \mathbb{1}_{\{\forall t \in (o,\beta);\, \omega \notin D\}}$$
$$\geqslant \frac{|D'|}{2\pi\beta}\left(1 - \frac{|\partial D'|}{|D'|}\sqrt{2\pi\beta}\, e^{-(d(\partial D, \partial D'))^2/2\beta}\right) \tag{5.10}$$

where at the last step we have used (5.8) and an explicit computation for the first term.

In our case we take $D' \subset D$ such that $|D'| = |D|/2$ and $d(\partial D', \partial D) \geqslant d/4$ and we get (3.21) for β sufficiently small.

6. APPENDIX 2

In this appendix we prove that (2.8) implies that there are at least two different Gibbs states. Let us recall that we have assumed that Λ is such that we can make a partition of Λ in boxes $\mathscr{D}$ of side $d = R/3\sqrt{2}$. Let us denote by Λ_d the set of all such boxes.

For any $\mathscr{D} \in \Lambda_d$ let us denote

$$\mathscr{B}(\mathscr{D}) \equiv \{(\underline{X}_\Lambda, \underline{Y}_\Lambda), \exists \omega_y^B \in \underline{Y}_\Lambda, \exists t \in [0, \beta] : \omega_y^B(t) \in \mathscr{D}\} \tag{6.1}$$

and

$$\mathscr{N}^B(\underline{X}_\Lambda, \underline{Y}_\Lambda) \equiv \sum_{\mathscr{D} \in \Lambda_d} \mathbb{1}_{\mathscr{B}(\mathscr{D})}(\underline{X}_\Lambda, \underline{Y}_\Lambda) \tag{6.2}$$

Given a contour γ, let $\sharp(\gamma)$ be the number of small boxes that are in the volume whose external boundary is γ. We have:

$$\begin{aligned}
\mu_{\Lambda,\beta,\zeta}(\mathscr{N}^B) &= \sum_{\mathscr{D} \in \Lambda_d} \mu_{\Lambda,\beta,\zeta}(\mathbb{1}_{\mathscr{B}(\mathscr{D})}) \\
&\leqslant \sum_{\mathscr{D} \in \Lambda_d} \sum_{\gamma \ni \mathscr{D}}^* (\gamma)\, \mu_{\Lambda,\beta,\zeta}(\gamma) \\
&\leqslant |\Lambda_d| \left(\sum_{n=3}^\infty n e^{-\zeta n/32\pi\beta} 3^{4n}\right)^2 \\
&\equiv |\Lambda_d|\, \tilde{\varphi}(\zeta/\beta)
\end{aligned} \tag{6.3}$$

where $\sum_{\gamma \ni \mathscr{D}}^{*}$ is the sum over all contours such that their internal volume contains $\mathscr{D}$ and $\tilde{\varphi}(\zeta/\beta)$ is a function independent of Λ that goes to zero when ζ/β goes to infinity. The step before the last is a standard computation for contours on a two dimensional lattice. So that,

$$\frac{\mu_{\Lambda, \beta, \zeta}(\mathscr{N}^{B})}{|\Lambda_d|} \leqslant \tilde{\varphi}(\zeta/\beta) \tag{6.4}$$

that is the mean of the density of the small boxes crossed by B-trajectories goes to zero, uniformly in Λ for $\zeta/\beta \uparrow \infty$.

To conclude the proof it remains to show that uniformly in Λ and in ζ/β the corresponding mean for the A-trajectories is bounded from below by a constant, different from 0.

Let T be the total number of small squares visited by at least a trajectory of any kind and define, for a given family of small squares $\mathscr{D}_1, ..., \mathscr{D}_k$ with $\mathscr{D}_i \neq \mathscr{D}_j$, for $1 \leqslant i < j \leqslant k$,

$$\begin{aligned}\mathscr{T}(\mathscr{D}_1, ..., \mathscr{D}_k) \equiv \Big\{ &(\underline{X_\Lambda}, \underline{Y_\Lambda}), \forall \omega \in (\underline{X_\Lambda}, \underline{Y_\Lambda}), \forall t \in [0, \beta], \omega(t) \in \bigcup_{i=1}^{k} \mathscr{D}_i, \\ &\forall 1 \leqslant j \leqslant k, \exists \omega, \exists \tau \in [0, \beta] : \omega(\tau) \in \mathscr{D}_j \Big\}\end{aligned} \tag{6.5}$$

and calling $\mathscr{V}_{11}(\mathscr{D})$ the square in Λ_d centered on $\mathscr{D}$ with side 11,

$$\begin{aligned}\mathscr{S}(\mathscr{D}_1, ..., \mathscr{D}_k) \equiv \Big\{ &(\underline{X_\Lambda}, \varnothing), \forall \mathscr{D} : \mathscr{D} \cap \bigcup_{i=1}^{k} \mathscr{V}_{11}(\mathscr{D}_i) = \varnothing, \\ &\exists \omega : \forall t \in [0, \beta], \omega(t) \in \mathscr{D} \\ &\forall \omega, \forall t \in [0, \beta], \omega(t) \cap \bigcup_{i=1}^{k} \mathscr{V}_{11}(\mathscr{D}_i) = \varnothing \Big\}\end{aligned} \tag{6.6}$$

Note that

$$\{T = k\} = \bigcup_{\mathscr{D}_1, ..., \mathscr{D}_k \subset \Lambda_d} \mathscr{T}(\mathscr{D}_1, ..., \mathscr{D}_k) \tag{6.7}$$

Let us call

$$\mathscr{T}_k \equiv \bigcup_{\mathscr{D}_1, ..., \mathscr{D}_k \subset \Lambda_d} \mathscr{T}(\mathscr{D}_1, ..., \mathscr{D}_k) \cup \mathscr{S}(\mathscr{D}_1, ..., \mathscr{D}_k) \tag{6.8}$$

and note that $\mathscr{T}_k \cap \mathscr{T}_{k'} = \varnothing$ if $k \neq k'$.

With these notations we have

$$
\begin{aligned}
Z_{\lambda,\beta,\zeta} &\geqslant \int dX_\Lambda\, dY_\Lambda\, h(X_\Lambda,Y_\Lambda)\, \mathbb{1}\left(\bigcup_{k=1}^{|\Lambda_d|/242} \mathscr{T}_k\right) \\
&= \sum_{k=1}^{|\Lambda_d|/242} \int dX_\Lambda\, dY_\Lambda\, h(X_\Lambda,Y_\Lambda)\, \mathbb{1}(\mathscr{T}_k) \\
&= \sum_{k=1}^{|\Lambda_d|/242} \sum_{\mathscr{D}_1,\ldots,\mathscr{D}_k \subset \Lambda_d} \int dX_\Lambda\, dY_\Lambda\, h(X_\Lambda, Y_\Lambda) \\
&\quad \times \mathbb{1}_{\mathscr{T}(\mathscr{D}_1,\ldots,\mathscr{D}_k)}(X_\Lambda, Y_\Lambda)\, \mathbb{1}_{\mathscr{S}(\mathscr{D}_1,\ldots,\mathscr{D}_k)}(X_\Lambda) \\
&\geqslant \sum_{k=1}^{|\Lambda_d|/242} \sum_{\mathscr{D}_1,\ldots,\mathscr{D}_k \subset \Lambda_d} \int dX_{\bigcup_{i=1}^k \mathscr{D}_i}\, dY_{\bigcup_{i=1}^k \mathscr{D}_i}\, h(X_{\bigcup_{i=1}^k \mathscr{D}_i}, Y_{\bigcup_{i=1}^k \mathscr{D}_i}) \\
&\quad \times \mathbb{1}_{\mathscr{T}(\mathscr{D}_1,\ldots,\mathscr{D}_k)}(X_{\bigcup_{i=1}^k \mathscr{D}_i}, Y_{\bigcup_{i=1}^k \mathscr{D}_i}) \prod_{\mathscr{D} \in \Lambda_d \setminus \bigcup_{i=1}^k \mathscr{V}_{11}(\mathscr{D}_i)} \int dX_{\mathscr{D}}\, \alpha_{\mathscr{D}}(X_{\mathscr{D}})
\end{aligned}
\tag{6.9}
$$

Recalling (5.9), using

$$
\left| \Lambda_d \setminus \bigcup_{i=1}^k \mathscr{V}_{11}(\mathscr{D}_i) \right| \geqslant |\Lambda_d| - 121k \tag{6.10}
$$

we have for ζ/β large enough,

$$
\prod_{\mathscr{D} \in \Lambda_d \setminus \bigcup_{i=1}^k \mathscr{V}_{11}(\mathscr{D}_i)} \int dX_{\mathscr{D}}\, \alpha_{\mathscr{D}}(X_{\mathscr{D}}) \geqslant (e^{\zeta d^2/8\pi\beta})^{|\Lambda_d| - 121k} \geqslant 2 \tag{6.11}
$$

therefore using the fact that

$$
\begin{aligned}
&Z_{\lambda,\beta,\zeta}[T=k] \\
&\quad = \sum_{\mathscr{D}_1,\ldots,\mathscr{D}_k \subset \Lambda_d} \int dX_{\bigcup_{i=1}^k \mathscr{D}_i}\, dY_{\bigcup_{i=1}^k \mathscr{D}_i}\, h(X_{\bigcup_{i=1}^k \mathscr{D}_i}, Y_{\bigcup_{i=1}^k \mathscr{D}_i}) \\
&\quad\quad \times \mathbb{1}_{\mathscr{T}(\mathscr{D}_1,\ldots,\mathscr{D}_k)}(X_{\bigcup_{i=1}^k \mathscr{D}_i}, Y_{\bigcup_{i=1}^k \mathscr{D}_i})
\end{aligned}
\tag{6.12}
$$

we get after a simple computation

$$
\mu_{\lambda,\beta,\zeta}\left[T \leqslant \frac{|\Lambda_d|}{242} \right] \leqslant \frac{1}{2} \tag{6.13}
$$

Therefore the mean number $\mu_{\lambda,\beta,\zeta}[T]$ of small boxes visited by A or B trajectories satisfies

$$\begin{aligned}\frac{\mu_{\lambda,\beta,\zeta}[T]}{|\Lambda_d|} &= \frac{1}{|\Lambda_d|}\sum_{k=1}^{\infty}\mu_{\lambda,\beta,\zeta}[T\geqslant k]\\ &\geqslant \frac{1}{|\Lambda_d|}\sum_{k=1}^{|L_d|/242}\mu_{\lambda,\beta,\zeta}[T\geqslant k]\\ &\geqslant \frac{1}{242}\mu_{\lambda,\beta,\zeta}\left[T\geqslant\frac{|\Lambda_d|}{242}\right]\\ &\geqslant \frac{1}{484}\end{aligned} \tag{6.14}$$

We have already shown that the mean density of small boxes visited by B-trajectories can be made arbitrarily small (see (6.4)). Now (6.4) and (6.14) imply that the mean density of small boxes visited by A-trajectories is bounded from below by a constant uniformly in Λ and ζ/β. Now changing the boundary conditions from A to B will exchange the previous mean densities which implies that the infinite volume limit Gibbs state is not unique.

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